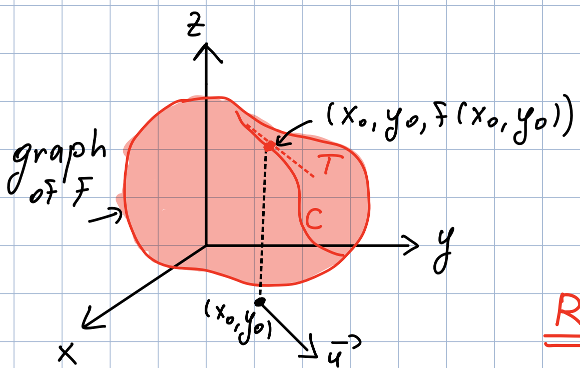
Last time:

$f(x,y)$ $\quad \vec{u} = \langle a, b \rangle$ ← unit vector

$$\underbrace{D_{\vec{u}} f(x_0, y_0)}_{\text{directional derivative of } f \text{ at } (x_0,y_0) \text{ in the direction of } \vec{u}} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$



$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)\, b}$$

Rmk: $\vec{u} = \langle 1, 0 \rangle \Rightarrow D_{\vec{u}} f = f_x$; $\vec{u} = \langle 0, 1 \rangle \Rightarrow D_{\vec{u}} f = f_y$

Gradient: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ $\qquad$ $D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$

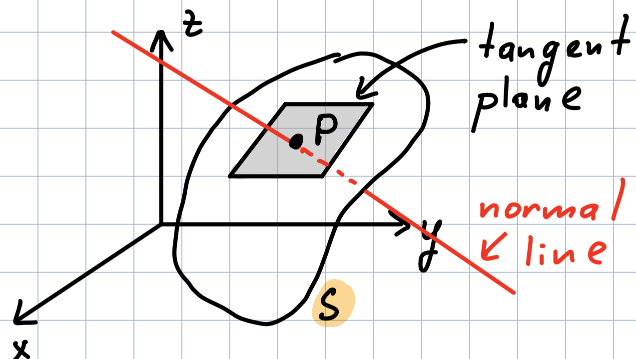
S: $F(x,y,z) = k$

– level surface

$P(x_0, y_0, z_0)$ point on S

the tangent plane to level surface $F(x,y,z) = k$ at $P(x_0, y_0, z_0)$:

$$\boxed{F_x(x_0,y_0,z_0)(x - x_0) + F_y(x_0,y_0,z_0)(y - y_0) + F_z(x_0,y_0,z_0)(z - z_0) = 0}$$



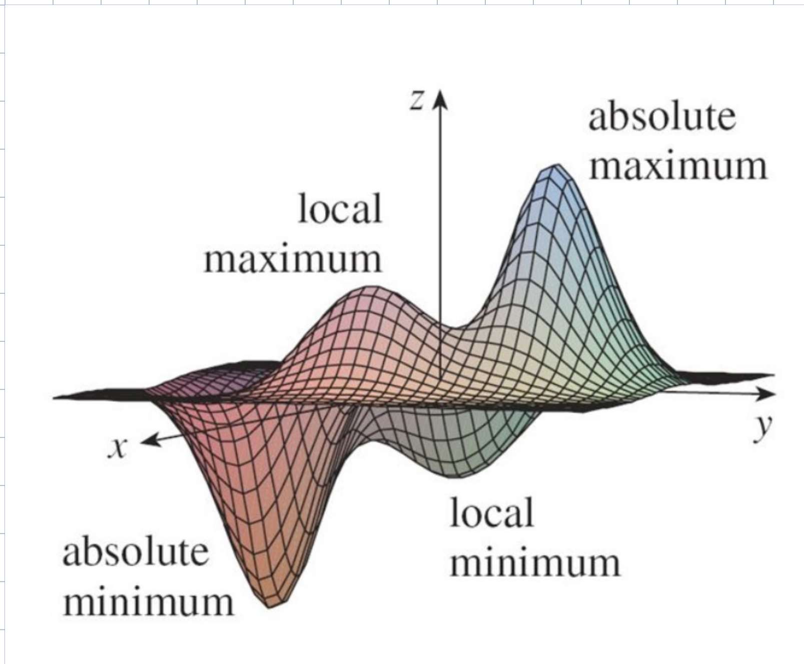
normal line to S at P:

line through P and orthogonal to tangent plane (i.e. it's direction vector is $\nabla F(x_0, y_0, z_0)$)

$$\Rightarrow \frac{x - x_0}{F_x(x_0,y_0,z_0)} = \frac{y - y_0}{F_y(x_0,y_0,z_0)} = \frac{z - z_0}{F_z(x_0,y_0,z_0)}$$

Local Extrema

$f(x,y)$



- A function $f(x,y)$ has a local maximum at (a,b) if $f(x,y) \leq f(a,b)$ when (x,y) is near (a,b).
- — // — local minimum at (a,b) if $f(x,y) \geq f(a,b)$ $\forall (x,y)$ near (a,b).

Thm: If f has a local maximum or minimum at (a,b) then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Rmk: i.e. $\nabla f(a,b) = 0$

Idea: Let $g(x) = f(x,b)$.

f has local max/min at $(a,b) \Rightarrow g$ has local max/min at a

$\Rightarrow$ [Fermat's thm] $g'(a) = 0$. Since $g'(a) = f_x(a,b) \Rightarrow f_x(a,b) = 0$.

• Recall that the equation of the tangent plane to the graph of $f(x,y)$ at the point $(x_0, y_0, \underbrace{f(x_0,y_0)}_{=z_0})$ is given by

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

• If $f(x,y)$ has a local max/min at (x_0,y_0) then $f_x(x_0,y_0)=0$ and $f_y(x_0,y_0)=0$

$\Rightarrow$ the tangent plane at local max/min is horizontal.

A point (a,b) is called a critical point of f if $f_x(a,b)=0$ and $f_y(a,b)=0$.

Rmk: local min/max is a critical point.

Ex: $f(x,y) = x^2+y^2-2x-3y+7$. Find all critical points.
Is it local max/min?

Sol: $\left.\begin{aligned} f_x(x,y) &= 2x-2 \\ f_y(x,y) &= 2y-3 \end{aligned}\right\}$ $f_x(1,\frac{3}{2}) = f_y(1,\frac{3}{2}) = 0$

The only critical point is $(1,\frac{3}{2})$.

Note that $f(x,y) = (x-1)^2 + (y-\frac{3}{2})^2 + \frac{15}{4} \geq \frac{15}{4}$ for all (x,y)

Therefore $f(1,\frac{3}{2})$ is a local minimum
(in fact it is the absolute minimum).

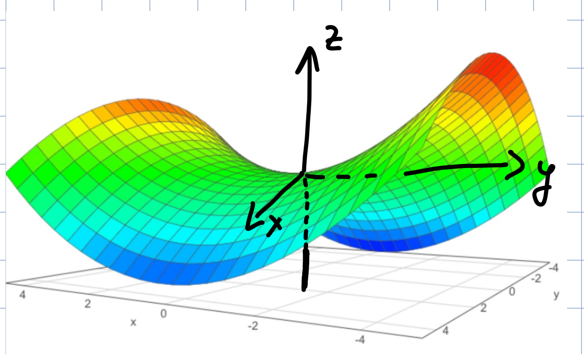
Ex: Find the extreme values of $f(x,y) = y^2 - x^2$

Sol: $f_x(x,y) = -2x$, $f_y(x,y) = 2y$ $\Big\}$ $(0,0)$ is the only critical point

at $(0,y)$: $f(0,y) \geq 0$

at $(x,0)$: $f(x,0) \leq 0$

$\Rightarrow$ $(0,0)$ is neither local max. nor local min.



$(0,0)$ is a saddle point of f

Critical points:
- local maximum
- local minimum
- saddle point

Second derivatives test

Suppose (a,b) is a critical point of f (i.e. $f_x(a,b)=0$ and $f_y(a,b)=0$).

Set $\mathcal{D}=\mathcal{D}(a,b)=f_{xx}(a,b)\,f_{yy}(a,b)-(f_{xy}(a,b))^2$

(a) If $\mathcal{D}>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ - local min.;

(b) If $\mathcal{D}>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ - local max.;

(c) If $\mathcal{D}<0$, then $f(a,b)$ is a saddle point of f.

Rmk: If $\mathcal{D}=0$, the test gives no information.

Rmk: $$\mathcal{D}=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=f_{xx}f_{yy}-(f_{xy})^2$$

Ex: Find local max/min. and saddle points of

$$f(x,y)=x^4+y^4-4xy+1.$$

Sol: 1) $f_x(x,y)=4x^3-4y$

$f_y(x,y)=4y^3-4x$

$$\begin{cases} x^3-y=0 & \Rightarrow \quad y=x^3 \\ y^3-x=0 & \Rightarrow \quad (x^3)^3-x=0 \end{cases}$$

$$0=x(x^8-1)=x(x^4+1)(x^4-1)=x\underbrace{(x^4+1)}_{>0}\underbrace{(x^2+1)}_{>0}(x^2-1)$$

$X=0, \quad X=1, \quad X=-1 \qquad y=x^3$

$\rightsquigarrow (0,0), \ (1,1), \ (-1,-1)$ are critical points

2) $D(x,y)$: $\quad f_{xx}=12x^2, \quad f_{xy}=-4, \quad f_{yy}=12y^2$

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = (12)^2x^2y^2-(-4)^2 = 144x^2y^2-16$$

3) $D(0,0)=-16 < 0 \Rightarrow f(0,0)$ - saddle point

$$\left.\begin{array}{l} D(1,1)=144-16=128>0 \\ f_{xx}(1,1)=12>0 \end{array}\right\} \quad f(1,1)=-1 \quad \text{- local min.}$$

$$\left.\begin{array}{l} D(-1,-1)=128>0 \\ f_{xx}(-1,-1)=12>0 \end{array}\right\} \quad f(-1,-1)=-1 \quad \text{- local min.}$$